The injectivity portion of combinatorial cuspidalization for FC-admissible outer automorphisms

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§1 Introduction

K: an NF or an MLF $\hookrightarrow$ $\overline{K}$: an alg closure of K

$G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$

(g, r): a pair of integers ≥ 0 s.t. $2g - 2 + r > 0$

C: a hyperbolic curve$_{/K}$ of type (g, r) (g: the genus, r: the $\sharp$ of cusps)

$\pi_1((-))$: the étale π_1 of $(-)$

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<u>Recall</u>: The homotopy ext seq

$$1 \longrightarrow \pi_1(C \times_K \overline{K}) \longrightarrow \pi_1(C) \longrightarrow G_K \longrightarrow 1$$

induces an outer representation

$$\rho : G_K \to \mathrm{Out}(\pi_1(C \times_K \overline{K})).$$

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Method: combinatorial anabelian geometry

- First, Mochizuki gave an alternative proof of Fact from the point of view of combinatorial anabelian geometry (cf. [CmbCsp]).
- Then Hoshi-Mochizuki proved Theorem 1 (cf. [NodNon]).

Idea

We know the injectivity of $G_K \to \mathrm{Out}(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\})$ (due to Belyĭ).
We want to show that $G_K \to \mathrm{Out}(\pi_1(C_{\overline{K}}))$ for any C. However,

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Observe: $\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}$ "$\subseteq$" the 3rd conf. space $(C_{\overline{K}})_3$ of $C_{\overline{K}}$.

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On the other hand,

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$\Longrightarrow$ We need to consider a certain subgp $\mathrm{Out}^{\mathrm{FC}}(\pi_1((C_{\overline{K}})_n))$.

k: an alg closed field of char 0

X: a hyperbolic curve$_{/k}$ of type (g, r)

$X_n \stackrel{\text{def}}{=} \{(x_1, \cdots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$ (the n-th conf. sp)

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In particular, the projections obtained by forgetting the last factors

$$X_n \to X_{n-1} \to \cdots \to X_2 \to X$$

induce a sequence of (outer) surjections

$$\Pi_n \overset{\text{def}}{=} \pi_1(X_n) \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1.$$

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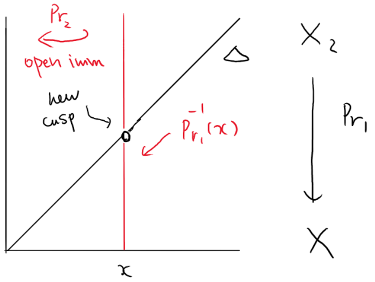
Write $K_m \overset{\text{def}}{=} \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$, $\Pi_0 \overset{\text{def}}{=} \{1\}$. Then we have

$$\{1\} = K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_1 \subseteq K_0 = \Pi_n.$$



Note: We have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_m/K_{m+1} & \longrightarrow & \Pi_n/K_{m+1} & \longrightarrow & \Pi_n/K_m & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 1 & \longrightarrow & \pi_1(Y) & \longrightarrow & \Pi_{m+1} & \longrightarrow & \Pi_m & \longrightarrow & 1 \end{array}$$

— where Y is a hyperbolic curve of type $(g, r+m)$.

Definition

$\alpha \in \mathrm{Aut}(\Pi_n)$ is F-admissible $\overset{\text{def}}{\Leftrightarrow}$ $\alpha(F) = F$ for every fiber subgroup $F \subseteq \Pi_n$ (i.e., the kernel of $\Pi_n \twoheadrightarrow \Pi_{n'}$ which arises from some projection $X_n \to X_{n'}$).

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$\alpha \in \mathrm{Aut}(\Pi_n)$ is C-admissible $\overset{\text{def}}{\Leftrightarrow}$

(i) $\alpha(K_m) = K_m$ $(0 \leq m \leq n)$;

(ii) $\alpha : K_m/K_{m+1} \overset{\sim}{\to} K_m/K_{m+1}$ induces a bijection between the set of cuspidal inertia subgps $\subseteq K_m/K_{m+1}$.

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$\alpha \in \mathrm{Aut}(\Pi_n)$ is FC-admissible $\overset{\text{def}}{\Leftrightarrow}$ α is F-admissible and C-admissible.

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \overset{\mathrm{def}}{=} \{\text{ FC-admissible automorphisms of } \Pi_n \}$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \overset{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)/\mathrm{Inn}(\Pi_n)$$

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Observe: $X_{n+1} \to X_n$ "forgetting the last factor" induces

$$\phi_n : \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n).$$

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Theorem 2 (Hoshi-Mochizuki)

ϕ_n *is injective for* $n \geq 1$.

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Remark:

- ϕ_n is bijective for $n \geq 4$.
- $\exists$ pro-l version of ϕ_n and similar results are known.

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$\boxed{\text{Theorem 2} \implies \text{Theorem 1}}$

Let $X \overset{\mathrm{def}}{=} C \times_K \overline{K}$, $k \overset{\mathrm{def}}{=} \overline{K}$

($\implies \pi_1(C \times_K \overline{K}) = \pi_1(X) = \Pi_1$)

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($\implies \pi_1(C \times_K \overline{K}) = \pi_1(X) = \Pi_1$)

Note: The outer rep'n $\rho : G_K \to \mathrm{Out}(\Pi_1)$ factors as

$$G_K \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \hookrightarrow \mathrm{Out}(\Pi_1).$$

Thus, to show that ρ is injective, it suffices to show that

$$G_K \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \text{ is injective.}$$

This follows from the commutativity of the diagram

$$\begin{tikzcd}
G_K \arrow[r] \arrow[d, equal] & \mathrm{Out}^{\mathrm{FC}}(\Pi_1) & \\
G_K \arrow[r] \arrow[d, equal] & \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \arrow[u, hook, "\phi_1"'] & \\
G_K \arrow[r] \arrow[rr, bend right, "\text{injective by Bely\u{\i}}"'] & \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \arrow[u, hook, "\phi_2"'] \arrow[r, dashed, "\exists"] & \mathrm{Out}(\pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}))
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Today, for simplicity, we consider the proof of the injectivity of ϕ_1.

$\Longrightarrow$ It suffices to verify the following proposition:

Proposition 1

Write

$$\mathrm{Aut}^{\mathrm{IFC}}(\Pi_2) \stackrel{\mathrm{def}}{=} \left\{ \alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2) \;\middle|\; \begin{tikzcd} & \Pi_1 \arrow[loop right, "\alpha_1 = \mathrm{id}"] \\ \Pi_2 \arrow[loop left, "\alpha"] \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & \\ & \Pi_1 \arrow[loop right, "\alpha_2 = \mathrm{id}"] \end{tikzcd} \right\},$$

$$\Xi \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p_1) \cap \mathrm{Ker}(p_2) \ (\subseteq \Pi_2).$$

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$$\Xi \overset{\mathrm{def}}{=} \mathrm{Ker}(p_1) \cap \mathrm{Ker}(p_2) \;\; (\subseteq \Pi_2).$$

Then the injection (cf. the center-freeness of Π_2*)*

$$\Xi \overset{\mathrm{conj.}}{\hookrightarrow} \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$$

is bijective.

$$\boxed{\text{Proposition 1} \implies \text{Theorem 2} \ \ (\phi_1 : \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1))}$$

Let $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ s.t.

$$\begin{tikzcd} & \Pi_1 \arrow[loop right, "\alpha_1"] & = \mathrm{Inn}(g_1) \\ \Pi_2 \arrow[loop left, "\alpha"] \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & \\ & \Pi_1 \arrow[loop right, "\alpha_2"] & \end{tikzcd}$$

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<u>Observe</u>: Let $\mathbb{D} \subseteq \Pi_2$ be a decomposition group assoc. to the diagonal $\subseteq X \times_k X$. Then it holds that

$$\alpha(\mathbb{D}) = \pi \cdot \mathbb{D} \cdot \pi^{-1} \quad (\pi \in \Pi_2).$$

(cf. our assumption that α is FC-admissible).

Thus, since

$$\begin{tikzcd} \mathbb{D} \arrow[r, hook] \arrow[d, two heads] & \Pi_2 \arrow[d, two heads, "(p_1, p_2)"] \\ \{(a,a)\} \arrow[r, hook] & \Pi_1 \times \Pi_1 \end{tikzcd}$$

we conclude that

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$\boxed{\text{Proof of Proposition 1}}$ $\cdots$ an application of CmbGC!

§2 Proof of Prop1 — the tripod case

Suppose that $\boxed{X \ = \ \mathbb{P}^1_k \setminus \{a, b, c\}}$.

Write $\Pi_{2/1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_2 \stackrel{p_1}{\twoheadrightarrow} \Pi_1)$. In particular, we have

$$1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_2 \xrightarrow{p_1} \Pi_1 \longrightarrow 1.$$

Let $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$. We want to show that α is a Ξ-inner.

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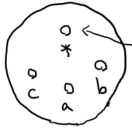
First, we consider the geom. generic fiber of $\mathrm{pr}_1 : X_2 \to X$.

$\Pi_{2/1} \cong \pi_1($ 

$)$ new cusp

Then since

- $p_2 : \Pi_2 \twoheadrightarrow \Pi_1$ is induced by the open immersion

,

- $\alpha_2 = \mathrm{id}$ (cf. $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$),
- α is C-admissible,

we conclude that α induces the identity permutation on the set of conjugacy classes of cuspidal inertia groups of $\Pi_{2/1}$.

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Therefore, replacing α by $\mathrm{Inn}(\xi^{-1}) \circ \alpha$, we may assume WLOG that

$$\alpha(I_a) \; = \; I_a.$$

Under this (additional) assumption, let us prove $\boxed{\alpha = \mathrm{id}}$.

Step 1 (group-theoretic argument)

Observe: We have an exact sequence

$$1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_2 \xrightarrow{p_1} \Pi_1 \longrightarrow 1.$$

Moreover, it holds that $\alpha_1 = \mathrm{id}$ (cf. $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$). Thus, since $\Pi_{2/1}$ is center-free, to verify $\alpha = \mathrm{id}$, it suffices to show that

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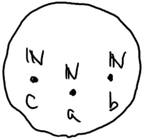
Step 2 (application of CmbGC)

$Z^{\log}$: a "natural" smooth log curve$_{/k}$ assoc. to X

$Z_2^{\log}$: the 2nd log configuration space of $Z^{\log}$

Note: $\Pi_n \xrightarrow{\sim} \pi_1^{\log}(Z_n^{\log})$.

$Z^{\log} =$



$\mathbb{P}^1_k$

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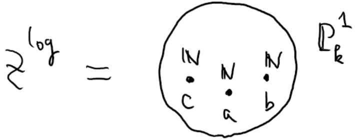

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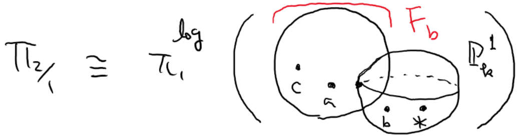

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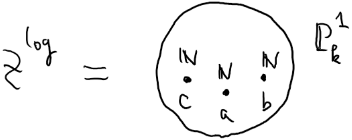

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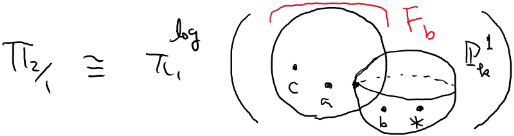

$\Pi_{F_b} \ \subseteq\ \Pi_{2/1}$: a unique (among its $\Pi_{2/1}$-cong.) verticial subgp
assoc. to F_b which contains (the fixed) I_a

Note: We have the following commutative diagram:

$$\begin{tikzcd} J_b \arrow[r, hook] \arrow[d, "\mathrm{id}"', "\wr"] & \Pi_1 \arrow[r] \arrow[d, "\alpha_1 = \mathrm{id}"', "\wr"] & \mathrm{Out}(\Pi_{2/1}) \arrow[d, "\mathrm{Out}(\alpha_{2/1})"', "\wr"] \\ J_b \arrow[r, hook] & \Pi_1 \arrow[r] & \mathrm{Out}(\Pi_{2/1}) \end{tikzcd}$$

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$\Longrightarrow$ $\alpha_{2/1}(\Pi_{F_b}) = \Pi_{F_b}$ (cf. the "uniqueness" of Π_{F_b})

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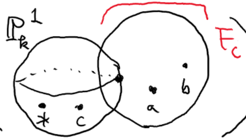
$\Pi_{2/1} \cong \pi_1^{\log}\Big($



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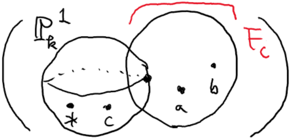
$\Pi_{2/1} \cong \pi_1^{\log}$ 

$\Pi_{F_c} \subseteq \Pi_{2/1}$: a unique (among its $\Pi_{2/1}$-cong.) verticial subgp assoc. to F_c which contains (the fixed) I_a

⋮

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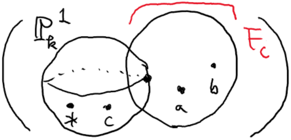

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$\vdots$

$\Longrightarrow \alpha_{2/1}(\Pi_{F_c}) = \Pi_{F_c}$

Note: $p_2 : \Pi_2 \twoheadrightarrow \Pi_1$ induces $\Pi_{F_b} \xrightarrow{\sim} \Pi_1$ and $\Pi_{F_c} \xrightarrow{\sim} \Pi_1$.

$\Longrightarrow$ $\alpha_{2/1}|_{\Pi_{F_b}} = \mathrm{id}, \quad \alpha_{2/1}|_{\Pi_{F_c}} = \mathrm{id}$ (cf. $\alpha_2 = \mathrm{id}$)

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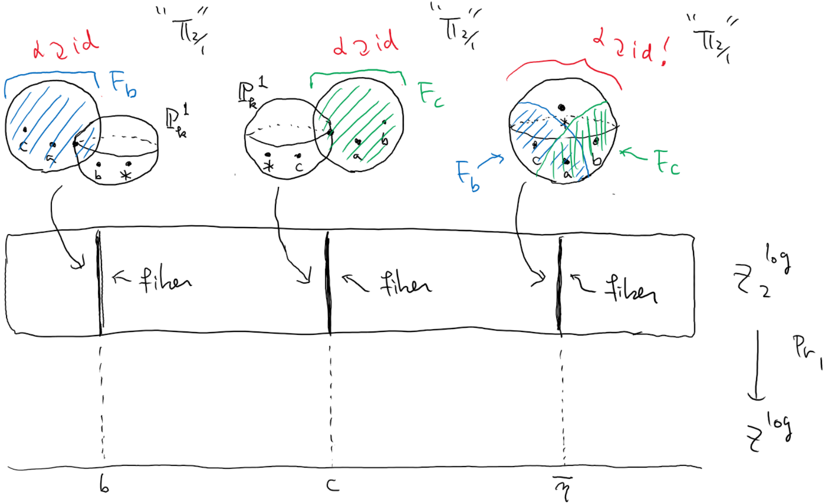
Step 3 (topological argument)

Since

$$\Pi_{2/1} \cong \varinjlim(\Pi_{F_b} \hookleftarrow I_a \hookrightarrow \Pi_{F_c})$$

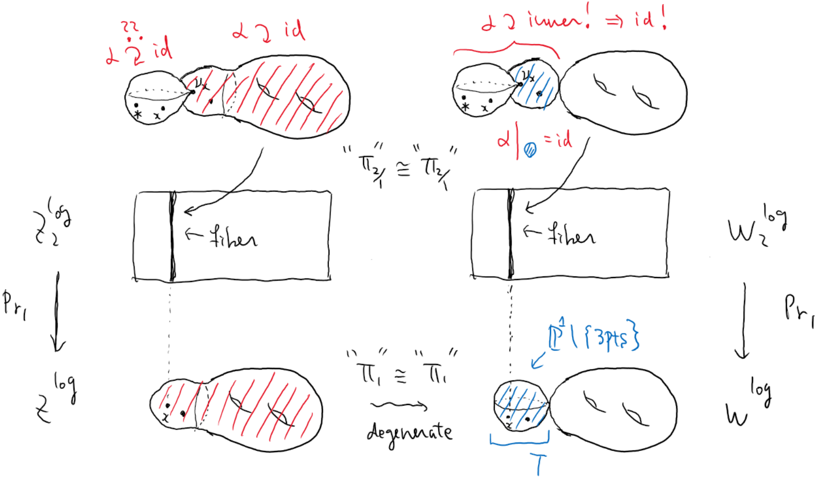
(cf. van Kampen), we conclude that $\alpha_{2/1} = \mathrm{id}$.

This completes the proof of the tripod case of Prop 1.



α ⊇ id
"Π_2/1"
F_b
P^1_k
a
b
c
*
α ⊇ id
"Π_2/1"
F_c
P^1_k
a
b
c
*
α ⊇ id!
"Π_2/1"
F_b
F_c
a
b
c
*
fiber
fiber
fiber
Z_2^log
Pr_1
Z^log
b
c
η̄

§3 Proof of Prop1 — the affine case



Suppose that $\boxed{X \text{ is affine}}$. For simplicity, we assume that $\boxed{r \geq 2}$.

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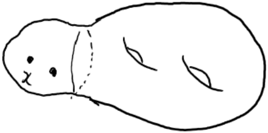

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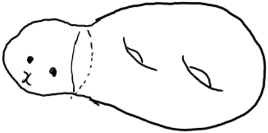
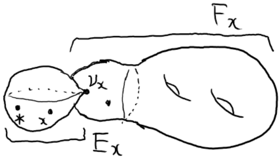
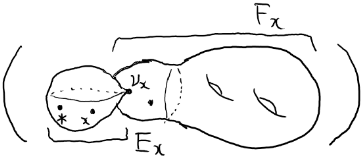
$Z^{\log} =$ 

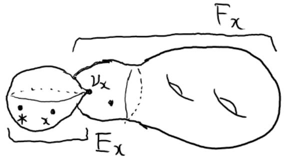
Let us consider the fiber of $\mathrm{pr}_1 : Z_2^{\log} \to Z^{\log}$ over x.

$\Pi_{2/1} \cong \pi_1^{\log}\Big($  $\Big)$

$\Pi_{2/1} \cong \pi_1^{\log}$


Fix: a (nodal) edge-like subgp $\Pi_{\nu_x} \subseteq \Pi_{2/1}$ assoc. to ν_x.

$\Pi_{E_x}, \ \Pi_{F_x} \ \subseteq \ \Pi_{2/1}$: two verticial subgp assoc. to E_x, F_x which contains (the fixed) Π_{ν_x}

$\Pi_{2/1} \cong \pi_1^{\log}\Big($  $\Big)$

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Then it follows from CmbGC that $\alpha_{2/1}$ is graphic, hence that

$$\alpha(\Pi_{\nu_x}) = \xi \cdot \Pi_{\nu_x} \cdot \xi^{-1} \quad (\xi \in \Pi_{2/1}).$$

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Therefore, replacing α by $\mathrm{Inn}(\xi^{-1}) \circ \alpha$, we may assume WLOG that

$$\alpha(\Pi_{\nu_x}) \ = \ \Pi_{\nu_x}.$$

Under this (additional) assumption, let us prove $\boxed{\alpha = \mathrm{id}}$.

Then since $\alpha_{2/1}$ is graphic, we conclude that

$$\alpha_{2/1}(\Pi_{E_x}),\ \ \alpha_{2/1}(\Pi_{F_x}) \text{ are verticial subgp } \supseteq \alpha_{2/1}(\Pi_{\nu_x}) = \Pi_{\nu_x}.$$

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$\Longrightarrow\ \alpha_{2/1}(\Pi_{E_x}) = \Pi_{E_x},\ \ \alpha_{2/1}(\Pi_{F_x}) = \Pi_{F_x}$

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Observe: We have an exact sequence

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Step 3 (topological argument)

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Then since

$$\Pi_{2/1} \cong \varinjlim(\Pi_{E_x} \hookleftarrow \Pi_{\nu_x} \hookrightarrow \Pi_{F_x})$$

(cf. van Kampen), to verify $\alpha_{2/1} = \mathrm{id}$, it suffices to show that

$$\boxed{\alpha_{2/1}|_{\Pi_{E_x}} = \mathrm{id}}.$$

To verify this, by applying

- deformation theory of stable log curves,
- specialization theorem of log étale fundamental groups,

we may replace "$Z^{\log}/k$" by

$W_2^{\log}$: the 2nd log configuration space of $W^{\log}$

Note: $\Pi_n \xrightarrow{\sim} \mathrm{Ker}(\pi_1^{\log}(W_n^{\log}) \twoheadrightarrow \pi_1^{\log}(S^{\log}))$.

Let us consider the fiber of $\mathrm{pr}_1 : W_2^{\log} \to W^{\log}$ over x.

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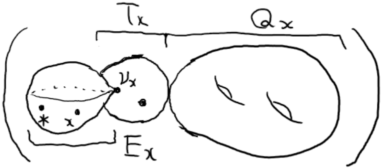

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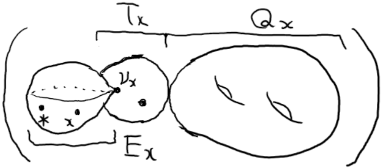

$\Longrightarrow \ \alpha_{2/1}|_{\Pi_{T_x}} \ = \ \mathrm{id}$ (cf. $\alpha_{2/1}|_{\Pi_{F_x}} = \mathrm{id}$)

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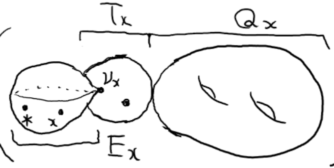
$\Pi_{2/1} \cong \pi_1^{\log}\Big($  $\Big)$

$\Longrightarrow\ \alpha_{2/1}|_{\Pi_{T_x}} \ = \ \mathrm{id} \quad$ (cf. $\alpha_{2/1}|_{\Pi_{F_x}} = \mathrm{id}$)

$T^{\log}$: a "natural" smooth log curve$_{/S^{\log}}$ assoc. to $T \setminus \{\text{marked pts, node}\}$

$T_2^{\log}$: the 2nd log configuration space of $Z^{\log}$

$\Pi_n^{\mathrm{tpd}} \ \stackrel{\mathrm{def}}{=}\ \mathrm{Ker}(\pi_1^{\log}(T_n^{\log}) \ \twoheadrightarrow\ \pi_1^{\log}(S^{\log}))$

In particular, we have

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{2/1} \arrow[r] & \Pi_2 \arrow[r, "p_1"] & \Pi_1 \arrow[r] & 1 \\
1 \arrow[r] & \Pi_{2/1}^{\mathrm{tpd}} \arrow[r] \arrow[u, hook] & \Pi_2^{\mathrm{tpd}} \arrow[r] \arrow[u, hook] & \Pi_1^{\mathrm{tpd}} \arrow[r] \arrow[u, hook] & 1.
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\end{tikzcd}
$$

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$$\alpha_{2/1}(\Pi_{2/1}^{\mathrm{tpd}}) \;=\; \Pi_{2/1}^{\mathrm{tpd}}$$

(cf. van Kampen).

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(cf. van Kampen). Moreover, one verifies that $\alpha_{2/1}|_{\Pi_{2/1}^{\mathrm{tpd}}}$ arises from

$$^{\exists}\alpha^{\mathrm{tpd}} \ \in \ \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2^{\mathrm{tpd}}) \ \xleftarrow{\sim} \ \Xi^{\mathrm{tpd}} \ \ (\subseteq \ \Pi_{2/1}^{\mathrm{tpd}})$$

(cf. §2). Thus, we conclude that $\alpha_{2/1}|_{\Pi_{2/1}^{\mathrm{tpd}}}$ is a $\Pi_{2/1}^{\mathrm{tpd}}$-inner.

Lemma 1

Let G be a profinite group, $H \subseteq G$ a closed subgroup, $\beta \in \mathrm{Inn}(G)$ s.t. $\beta|_H = \mathrm{id}$. Suppose that

- $N_G(H) = H$;
- *H is center-free.*

Then $\beta = \mathrm{id}$.

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We would like to apply this Lemma, to the present situation, by taking

$$\text{“}G\text{” to be } \Pi_{2/1}^{\mathrm{tpd}}, \quad \text{“}H\text{” to be } \Pi_{T_x}, \quad \text{“}\beta\text{” to be } \alpha_{2/1}|_{\Pi_{2/1}^{\mathrm{tpd}}}.$$

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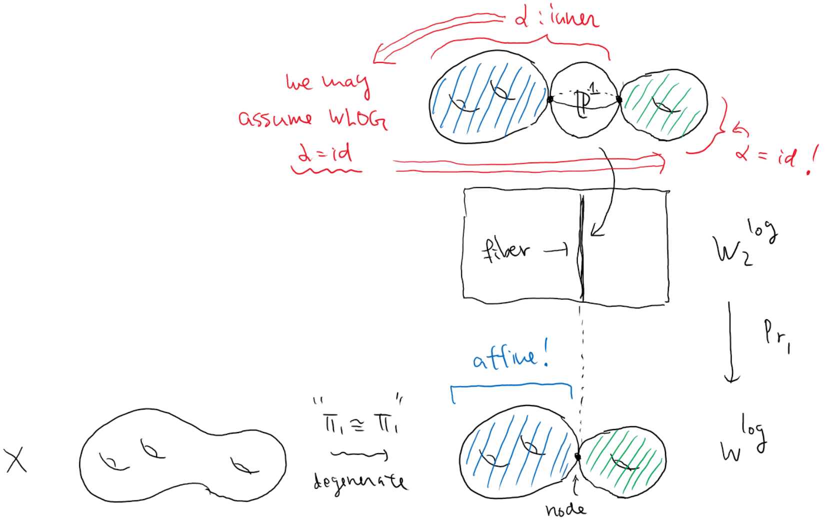
We would like to apply this Lemma, to the present situation, by taking

$$\text{“}G\text{” to be } \Pi_{2/1}^{\mathrm{tpd}}, \quad \text{“}H\text{” to be } \Pi_{T_x}, \quad \text{“}\beta\text{” to be } \alpha_{2/1}|_{\Pi_{2/1}^{\mathrm{tpd}}}.$$

Therefore, we conclude that $\alpha_{2/1}|_{\Pi_{2/1}^{\mathrm{tpd}}} = \mathrm{id}$, hence that $\alpha_{2/1}|_{\Pi_{E_x}} = \mathrm{id}$.

This completes the proof of the affine case of Prop 1.

§4 Proof of Prop1 — the proper case



$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: an outer representation of PSC-type

$\widetilde{\mathcal{G}} \to \mathcal{G}$: a universal covering

Π_I: the profinite gp obt'd by "pulling back the exact sequence"

$$1 \longrightarrow \Pi_{\mathcal{G}} \xrightarrow{\text{conj.}} \mathrm{Aut}(\Pi_{\mathcal{G}}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}) \longrightarrow 1$$

by the composite $I \overset{\rho_I}{\to} \mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$

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Definition

If $\tilde{z} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ or $\mathrm{Node}(\widetilde{\mathcal{G}})$, then we shall write

$$I_{\tilde{z}} \stackrel{\text{def}}{=} Z_{\Pi_I}(\Pi_{\tilde{z}}) \subseteq D_{\tilde{z}} \stackrel{\text{def}}{=} N_{\Pi_I}(\Pi_{\tilde{z}}).$$

Definition

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$: an outer representation of PSC-type

ρ_I is of NN-type $\overset{\text{def}}{\Leftrightarrow}$

(1) $I \cong \widehat{\mathbb{Z}}$.

(2) For every $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the image of $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is open.

(3) For every $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, the natural inclusions $I_{\tilde{v}_1}, I_{\tilde{v}_2} \subseteq I_{\tilde{e}}$ — where $\tilde{e}$ abuts to $\tilde{v}_1, \tilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ — induces an open injection $I_{\tilde{v}_1} \times I_{\tilde{v}_2} \hookrightarrow I_{\tilde{e}}$.

Theorem 3 (CmbGC for outer rep'ns of NN-type)

$\rho_I : I \to \mathrm{Aut}(\mathcal{G})$, $\rho_J : J \to \mathrm{Aut}(\mathcal{H})$*: outer rep'ns of PSC-type*

$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$*: an isom. which fits into a comm. diag.*

$$\begin{tikzcd} I \arrow[r, "\rho_I"] \arrow[d, "\wr"] & \mathrm{Aut}(\mathcal{G}) \arrow[r, hook] & \mathrm{Out}(\Pi_{\mathcal{G}}) \arrow[d, "\wr\ \mathrm{Out}(\alpha)"] \\ J \arrow[r, "\rho_J"] & \mathrm{Aut}(\mathcal{H}) \arrow[r, hook] & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{tikzcd}$$

— *where* $I \xrightarrow{\sim} J$ *is an isomorphism. Suppose that*

(i) ρ_I, ρ_J *are of NN-type.*

(ii) $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$ *and* α *is group-theoretically cuspidal.*

Then α *is graphic.*